Some Continuous Aproximation Theorems

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Abstract

We consider the following problem: When X is a topological space and V a normed space, how well can an arbitrary bounded function $f : X \to V$ be approximated by a continuous one?

We introduce a natural lower bound and establish several necessary and sufficient conditions for this bound to be achieved on particular pairs of X and V.

1 Motivating theorem

Let X be a topological space, V a normed space and $f : X \to V$ bounded. We are interested in studying how "far from being continuous" f is. In particular, what is the value of $d(f, C(X, V))$ in the uniform norm?

The following proposition will lead us to a lower bound on the answer:

Proposition 1. *Let $f, g : X \to V$ with g continuous. For every $x \in X$ and $\epsilon > 0$ there exists an open set U and a point $v \in V$ such $f(U) \subseteq B(v, ||f - g|| + \epsilon)$*

Proof. g is continuous, so we can find $U \ni x$ with $g(U) \subseteq B(g(x), \epsilon)$. Then for $y \in U$, $||f(y) - g(x)|| \leq ||f(y) - g(y)|| + ||g(y) - g(x)|| < ||f - g|| + \epsilon$. Hence $f(U) \subseteq B(g(x), ||f - g|| + \epsilon)$ as desired. □ □

This leads to us introducing the follow definitions:

Definition 2. *Let V be a normed space and $A \subseteq V$ bounded. Define*

$$r(A) = \inf\{r : \exists v, A \subseteq B(v, r)\}$$

Now let X be a topological space and $f : X \to V$. For $x \in X$, define

$$\rho_x(f) = \inf\{r(f(U)) : U \ni x, U \text{ open}\}$$

$$\rho(f) = \sup_x \rho_x(f)$$

These definitions are very close to the classic definition of the oscillation for a function. Indeed, for $V = \mathbb{R}$ we have $r(A) = \frac{1}{2}\text{diam}(A)$ and $\rho(f) = \frac{1}{2}\omega(f)$, but this does not hold for normed spaces in general.

Lemma 3.

$$d(f, C(X, V)) \geq \rho(f)$$

Proof. Let $\epsilon > 0$ and let g be continuous with $||f - g|| < d(f, C(X, V)) + \frac{1}{2}\epsilon$.

Let $x \in X$. By 1 we have that there exists U with $r(f(U)) \leq ||f - g|| + \frac{1}{2}\epsilon \leq d(f, C(X, V)) + \epsilon$.

Hence $\rho_x(f) \leq d(f, C(X, V))$ and thus $\rho(f) \leq d(f, C(X, V))$. ☐

Additionally, it's relatively easy to see the following lemma:

Lemma 4. *f is continuous iff $\rho(f) = 0$.*

Proof. It follows from the preceding lemma that if f is continuous then $\rho(f) = 0$.

If $\rho(f) = 0$, let $x \in X$ and $\epsilon > 0$. Pick $U \ni x$ such that $f(U) \subseteq B(v, \frac{1}{2}\epsilon$. Then for $y \in U$, $||f(x) - f(y)|| \leq ||f(x) - v|| + ||f(y) - v|| < \epsilon$.

☐

The question we want to answer: For what pairs X, V is this inequality always an equality? i.e. when is local information about the existence of continuous approximations enough to guarantee the global existence of such.

We introduce the following definition:

Definition 5. *We say X has optimal continuous approximations into V, or $OCA(X, V)$, if for all bounded $f : X \to V$ we have $\rho(f) = d(f, C(X, V))$. If X is OCA into all normed spaces V we say X is OCA.*

We will establish two theorems to determine $OCA(X, V)$ and then study a natural class of examples for which these theorems don't give the full picture.

We will need the following lemma:

Lemma 6. *Let X be a topological space, V a normed space, $f : X \to V$. Let $T > 0$ be a real number.*

If $\{U_a : a \in A\}$ is an open cover of X such that $f(U_a) \subseteq B(v_a, T)$, and p_a is a partition of unity subordinate to $\{U_a\}$, then $||g - f|| \leq T$, where

$$g(x) = \sum p_a(x) v_a$$

In particular given the existence of such a cover with a partition of unity, $d(f, C(X, V)) \leq T$.

Proof. Let $x \in X$. Consider those U_a containing x. For each of these U_a we have $v_a \in B(f(x), T)$. Therefore any convex combination of them is also in $B(f(x), T)$, as balls are convex. But $g(x)$ is a convex combination of these v_a, thus is in $B(f(x), T)$. Thus $||g(x) - f(x)|| < T$. Hence $||g - f|| \leq T$. ☐

Which immediately gives us our first positive result:

Theorem 7. *Any paracompact space is OCA.*

Proof. Let X be paracompact, V a normed space, $f : X \to V$ bounded.

If $T > \rho(f)$ then for every $x \in X$ we can find open sets $U_x \ni x$ such that $r(f(U_x)) < T$. X is paracompact, so this open cover has a subordinate partition of unity, and thus by 6 we have $d(f, C(X, V)) \leq T$. Taking infimum over T we thus have $d(f, C(X, V)) \leq \rho(f)$ as desired. ☐

By being a little more clever about our choice of open cover we can prove a finer grained result, which is the first of our two main theorems:

Theorem 8. *Let κ be a cardinal, X a topological space with the property that every open cover of fewer than κ open sets has a subordinate partition of unity. Let V be a normed space and $f : X \to V$ be such that for every $\epsilon > 0$, $\overline{f(X)}$ has an ϵ net of size $< \kappa$. Then $d(f, C(X, V)) = \rho(f)$.*

Proof. This proof is adapted from that for [3] (we could use the main theorem of that paper, but preferred to prove this directly as it is slightly more straightforward and lets us be explicit about the role of κ).

The proof will proceed as with the preceding one. We simply need to be slightly more clever with our construction of the open cover we use.

Let $\epsilon > 0$ and $\{v_\alpha : \alpha < \kappa\}$ be an $\frac{1}{2}\epsilon$ net for $\overline{f(X)}$. Define

$$U_\alpha = \bigcup \{U \subseteq X : U \text{ open}, f(U) \subseteq B(v_\alpha, \rho(f) + \epsilon)\}$$

U_α is open as it is a union of open sets. Further the U_α form an open cover of X: Given $x \in X$ we can find $U \ni x$ such that $f(U) \subseteq B(v, \rho(f) + \frac{1}{2}\epsilon)$. We can then find some v_α such that $||v_\alpha - v|| < \frac{1}{2}\epsilon$. Then $f(U) \subseteq B(v_\alpha, \rho(f) + \epsilon)$ and so $x \in U_\alpha$.

Now, the U_α constitute an open cover of X of size $< \kappa$, so have a subordinate partition of unity, by hypothesis, say g_α. 6 does the rest. ☐

This has a bunch of immediate corollaries:

Corollary 8.1. *If V is a normed space with a dense subset of size κ and X is a topological space such that every open cover of size κ has a subordinate partition of unity then $OCA(X, V)$.*

Corollary 8.2. *If X is normal and V finite dimensional then $OCA(X, V)$.*

Corollary 8.3. *If X is countably paracompact and V separable then $OCA(X, V)$.*

In fact the latter two of these corollaries are actually equivalences:

Theorem 9. *Let X be a topological space. The following are equivalent:*

1. *X is normal*
2. *$OCA(X, V)$ for any finite dimensional V.*
3. *$OCA(X, \mathbb{R})$*

Proof. We have already proven the forward directions. We need only show that $OCA(X, \mathbb{R})$ implies X is normal.

Let E, F be disjoint closed sets. Define $f : X \to \mathbb{R}$ by $f(x) = -1$ if $x \in E$, $f(x) = 1$ if $x \in F$ and else $f(x) = 0$.

It is easy to see that $r(E^c), r(F^c) \leq \frac{1}{2}$ and therefore that $\rho(f) \leq \frac{1}{2}$. Suppose g is continuous with $||g - f|| \leq \frac{3}{4}$. Then $g^{-1}((-\infty, 0))$ and $g^{-1}((0, \infty))$ are disjoint open sets containing E and F respectively. ☐

We can prove a similar result for separable vector spaces:

Theorem 10. *Let X be a topological space. The following are equivalent:*

1. *X is countably paracompact*
2. *X is $OCA(V)$ for any separable V*
3. *X is $OCA(c_0)$.*

Proof. Again it suffices to prove that $3 \implies 1$.

Let $\{U_n : n \in \mathbb{N}\}$ be a countable open cover of X.

Define $f : X \to c_0$ by $f(x) = e_n$ if $n = \min\{m : x \in U_m\}$.

We wish to compute $\rho(f)$. For any $x \in X$ we can find an n with $x \in U_n$ and then $f(U_n) \subseteq \{e_m : m \leq n\}$, so $r(f(U_n)) \leq \frac{1}{2}$.

Thus, $\rho(f) \leq \frac{1}{2}$. We have assumed X is $OCA(c_0)$, so we can find a continuous $g : X \to c_0$ with $||f - g|| < \frac{3}{4}$.

Let $V_n = \{x \in X : g(x)_n > \frac{1}{4}\}$. Each V_n is an open subset of X. Let $W_n = U_n \cap V_n$.

For any $x \in X$ we can let $n = \min\{m : x \in U_m\}$. Thus $f(x)_n = 1$ and $g(x)_n > \frac{1}{4}$. Hence $x \in W_n$. So, $\{W_n\}$ is an open cover of X.

Further, $g(x)_n$ is eventually $< \frac{1}{8}$, so $B(g(x), \frac{1}{8})$ intersects only finitely many of the sets $\{y \in c_0 : y_n > \frac{1}{4}\}$. Thus, $g^{-1}(B(g(x), \frac{1}{8}))$ is an open subset of X which contains x and intersects only finitely many of the V_n, and so certainly only intersects finitely many of the W_n.

Thus in fact $\{W_n\}$ is a locally finite cover of X, and it is a refinement of $\{U_n\}$.

Thus X is countably paracompact. $\square$

2 OCA and products

Let X, Y be topological spaces and V a vector space. For $f : X \to C(Y, V)$ define $\hat{f} : X \times Y \to V$ by $\hat{f}(x, y) = f(x)(y)$. Clearly the map $f \to \hat{f}$ is linear and norm preserving.

Lemma 11. *Let $f : X \to C(Y, V)$. $\rho(f) \geq \rho(\hat{f})$.*

Proof. Let $x \in X$, $y \in Y$ and $\epsilon > 0$.

Find open $U \ni x$ be such that $f(U) \subseteq B(g, \rho(f) + \frac{1}{2}\epsilon)$ for some continuous g.

Now find open $V \ni y$ such that for $y' \in V$ we have $||g(y') - g(y)|| < \frac{1}{2}\epsilon$.

For $(x', y') \in U \times V$ we have

$$||f(x')(y') - g(y)|| \leq ||f(x')(y') - g(y')|| + ||g(y) - g(y')|| \leq \rho(f) + \epsilon$$

so we have a neighbourhood $U \times V$ of (x, y) such that $f(U \times V) \subseteq B(g(y), \rho(f) + \epsilon$. Thus, $r(f(U \times V)) \leq \rho(f) + \epsilon$. Letting $\epsilon \to 0$ we have that $\rho_{(x,y)}(\hat{f}) \leq \rho(f)$ and thus $\rho(\hat{f}) \leq \rho(f)$. $\square$

Corollary 11.1. *If $f : X \to C(Y, V)$ is continuous then so is $\hat{f}$*

Lemma 12. *Let K be compact and $f : X \to C(K, V)$. $\rho(f) = \rho(\hat{f})$.*

Proof. It suffices to show that $\rho(f) \leq \rho(\hat{f})$.

Let $T > \rho(\hat{f})$. Then we can cover $X \times K$ with sets of the form $U \times V$ such that $\hat{f}(U \times V) \subseteq B(v, T)$ for some v.

Fix $x \in X$ and consider only the sets $U \times V$ where $x \in U$. The V from these sets form an open cover of K, and thus have a finite subcover.

Thus we can find $U_1, ..., U_n, V_1, ..., V_n$ such that the V_i cover K, all the U_i contain x and $\hat{f}(U_i \times V_i) \subseteq B(c_i, T)$.

Now let p_i be a subordinate partition of unity to the V_i. Let $g(y) = \sum p_i(y)c_i$. By 6 we have that for any $w \in \bigcap U_i$, the function $f(w)$ is within T of g, as $f(w)(V_i) \subseteq B(c_i, T)$.

But there are only finitely many U_i, so $U = \bigcap U_i$ is open and a neighbourhood of x. We thus have that $f(U) \subseteq B(g, T)$. Hence $\rho_x(f) < T$. Taking sup over x gives $\rho(f) \leq T$, and inf over T yields the desired conclusion. ☐

Corollary 12.1. *$f : X \to C(K, V)$ is continuous iff $\hat{f}$ is continuous.*

Theorem 13. *Let K be compact. $OCA(X \times K, V) \implies OCA(X, C(K, V))$.*

Proof. Let $f : X \to C(K, V)$ and $\epsilon > 0$. By the preceding lemma, $\rho(\hat{f}) = \rho(f)$. Because $OCA(X \times K, V)$ there is a continuous g with $||\hat{f} - g|| < \rho(f) + \epsilon$. We may find a unique h with $g = \hat{h}$, and again by the preceding lemma this must be continuous. Hence $||f - h|| = ||\hat{f} - g|| < \rho(f) + \epsilon$. Hence letting $\epsilon \to 0$ we have $d(f, C(K, V)) = \rho(f)$. ☐

Corollary 13.1. *If K is compact and $X \times K$ is normal then $OCA(X, C(K))$.*

The converse of this corollary is false. We'll now consider some of the behaviour of ω_1, from which several counterexamples to this will fall out.

3 Behaviour of ω_1

Definition 14. *$c_0(\omega_1)$ is the set of bounded continuous functions from $\omega_1 \to \mathbb{R}$ that are eventually 0.*

We will show that $\neg OCA(\omega_1, c_0(\omega_1))$

Lemma 15. *Let $K \subseteq c_0(\omega_1)$ be separable. There exists $\alpha < \omega_1$ such that for all $\beta > \alpha$ and $f \in K$, $f(\beta) = 0$.*

Proof. Let $\{x_n\}$ be a countable dense subset of K. Then there exists α_n such that for $\beta > \alpha_n$ we have $x_n(\beta) = 0$. Let $\alpha = \sup \alpha_n$. Then for $\beta > \alpha$ we have $x_n(\beta) = 0$. Thus $x \to x(\beta)$ is a continuous function which is 0 on a dense subset of K and so on K. ☐

Theorem 16. *$\neg OCA(\omega_1, c_0(\omega_1))$.*

Proof. Define $f : \omega_1 \to c_0(\omega_1)$ as $f(\alpha) = 1_{[0,\alpha]}$. This has $\rho(f) \leq \frac{1}{2}$, as $f([0, \alpha]) \subseteq B(\frac{1}{2} 1_{[0,\alpha]}, \frac{1}{2})$.

Now let $g : \omega_1 \to c_0(\omega_1)$ with $||f - g|| \leq \frac{1}{2} + \epsilon$ be continuous. g is continuous and ω_1 is countably compact, so $g(\omega_1)$ is countably compact. But $c_0(\omega_1)$ is a metric space, so countably compact subspaces are compact. Therefore $g(\omega_1)$ is compact and thus separable.

Hence by **??** we can find some α such that for all $x \in \omega_1$, $g(x)(\alpha) = 0$. But this means that $|g(\alpha)(\alpha) - f(\alpha)(\alpha)| = 1$ and so $||f - g|| \geq 1$. Thus $d(f, C(\omega_1, c_0(\omega_1))) \geq 1$. ☐

In fact, this sort of example exactly characterises the only way in which $OCA(\omega_1, V)$ can fail .

Definition 17. *Let* V *be a normed space.* $\{v_\alpha : \alpha < \omega_1\}$ *is a bad sequence if*

$$\forall \beta < \omega_1, r(\{v_\alpha : \alpha \leq \beta\}) \leq 1$$

$$\inf_{\beta < \omega_1} r(\{v_\alpha : \alpha > \beta\}) > 1$$

Theorem 18. *Let* V *be a normed space.* $OCA(\omega_1, V)$ *iff* V *has no bad sequences.*

Proof. Suppose v_α is a bad sequence in V. Then as in the proof of 16 example for $c_0(\omega_1)$ the function $f : \alpha \mapsto v_\alpha$ has $\rho(f) \leq 1$ and $d(f, C(\omega_1, C(\omega_1))) > 1$.

Conversely, suppose V has no bad sequences and let $f : \omega_1 \to V$ be bounded.

Fix $\epsilon > 0$. We will show that there exists $\alpha < \omega_1$ such that $r(f([\alpha, \omega_1))) \leq \rho(f) + \epsilon$. This will then suffice to find a continuous approximation: Say c forms a center for this set. $[0, \alpha]$ is compact, so we can find a continuous approximation, g, to f on there, and we then define $g(\beta) = c$ for $\beta > \alpha$.

By definition of ρ, for every $\alpha < \omega_1$ we may find a neighbourhood U of α such that $r(f(U)) < \rho(f) + \epsilon$. In particular if $\alpha > 0$, for some $\beta < \alpha$, $r(f((\beta, \alpha]))) < \rho(f) + \epsilon$.

Define $h : \omega_1 \backslash \{0\} \to \omega_1$ by letting $h(\alpha)$ be the minimum such β. Clearly $h(\alpha) < \alpha$, so h is regressive.

But then by Fodor's lemma h is constant on some unbounded set. Hence we may find some β and an unbounded set F such that for $\alpha \in F$, $r(f((\beta, \alpha])) < \rho(f) + \epsilon$

Let $v_\alpha = \frac{f(\beta+\alpha)}{\rho(f)+\epsilon}$. Then each initial segment has radius ≤ 1. Because V has no bad sequences, there is no $C > 1$ such that every tail has radius $\geq C$. In particular we can find some tail, say after γ, with radius $\leq 1 + \epsilon$.

But then we must have $r(f([\beta + \gamma, \omega_1))) \leq (\rho(f) + \epsilon)(1 + \epsilon)$. So $d(f, C(X, V)) \leq (\rho(f) + \epsilon)(1 + \epsilon)$.

Letting $\epsilon \to 0$ the result is proved. □

There's one easy way to ensure the non-existence of bad sequences:

Definition 19. *Let* κ *be a cardinal. A normed space* V *has* κ*-determined radii if*

$$\forall A \subseteq V . r(A) = \sup\{r(B) : B \subseteq A, |B| < \kappa\}$$

Evidently if V is κ determined then it is κ' determined for any $\kappa' > \kappa$.

Lemma 20. *If* V *has* ω_1 *determined radii then it does not contain a bad sequence.*

Proof. Let v_α be such that $r(\{v_\alpha : \alpha < \beta\}) \leq 1$.

Every countable subset of $\{v_\alpha\}$ is contained in one of these initial segments, therefore has radius $\leq r$. Since V has ω_1 determined radii, $r(\{v_\alpha\})$ is the supremum of the radii of its countable subsets and thus ≤ 1. ☐

Two important classes of spaces have ω determined radii.

Theorem 21. *Any reflexive space has ω determined radii.*

Proof. Let $A \subseteq V$ be bounded. Let $T < r(A)$. Then

$$\bigcap_{a \in A} \overline{B}(a, T) = \emptyset$$

as any element of the intersection would witness that $r(A) \leq T$.

But the closed unit ball of a reflexive space is weakly compact, so this is an intersection of closed non-empty sets in a compact topology. Therefore some finite subset of it has an empty intersection. Therefore that finite subset must have radius $\geq T$. But T was arbitrary $< r(A)$, therefore

$$r(A) = \sup\{r(B) : B \subseteq A, B \text{finite}\}$$

☐

Lemma 22. *If V has the property that $\forall A \subseteq V . r(A) = t\text{diam}(A)$ for some constant t then it has ω determined radii.*

Proof. This follows immediately from the fact that the diameter is defined by finite subsets (indeed subsets of size 2). ☐

Corollary 22.1. *$l^\infty(A)$ has ω-determined radii for any set A.*

Unfortunately neither of these cases cover $C(\omega_1)$. We will need some machinery for calculating radius in $C(X)$ (with thanks to with thanks to [1] for the main idea of this proof).

Definition 23. *Let X be a topological space and $\mathcal{F} \subseteq C(X)$ be bounded. Define*

$$\mathcal{F}^*(x) = \inf_{U \ni x} \sup_{f \in \mathcal{F}, y \in U} f(Y)$$

$$\mathcal{F}_*(x) = \sup_{U \ni x} \inf_{f \in \mathcal{F}, y \in U} f(Y)$$

Proposition 24. *$\mathcal{F}^*$ is upper semicontinuous and $\mathcal{F}_*$ is lower semicontinuous.*

Proof. Suppose $\mathcal{F}^*(x) < t$. Then there is some $U \ni x$ such that $\sup \mathcal{F}(U) < t$. Then for $y \in U$ we must have $\mathcal{F}^*(y) < t$. Hence $\{x : \mathcal{F}^*(x) < t\}$ is open, and so $\mathcal{F}^*$ is upper semi-continuous. The other case follows similarly. ☐

Lemma 25. *$\mathcal{F}^* - \mathcal{F}_* \leq 2r(\mathcal{F})$.*

Proof. Let $R = r(\mathcal{F})$.

Fix $\epsilon > 0$ and $x \in X$. Suppose $\mathcal{F} \subseteq B(f, R + \epsilon)$.

Now find $U \ni x$ such that for $y \in U$, $|f(y) - f(x)| < \epsilon$.

Then for $y \in U$ and $f \in \mathcal{F}$ we must have $|f(y) - g(x)| \leq |f(y) - g(y)| + |g(x) - g(y)| \leq R + 2\epsilon$.

Thus we have $\mathcal{F}^*(x) \leq \sup \mathcal{F}(U) \leq g(x) + R + 2\epsilon$ and $\mathcal{F}_*(x) \geq g(x) - R - 2\epsilon$. Hence $\mathcal{F}^*(x) - \mathcal{F}_*(x) \leq 2R + 4\epsilon$. Letting $\epsilon \to 0$ the result is proved. ☐

Lemma 26. *If X is normal then $r(\mathcal{F}) = \frac{1}{2}\sup_x(\mathcal{F}^*(x) - \mathcal{F}_*(x))$*

Proof. Let

$$R = \frac{1}{2}\sup_x(\mathcal{F}^*(x) - \mathcal{F}_*(x))$$

By the preceding lemma we have $r(\mathcal{F}) \geq R$, so it suffices to show that $r(\mathcal{F}) \leq R$.

Evidently we have

$$\mathcal{F}^* \leq \mathcal{F}_* + 2R$$

and so

$$\mathcal{F}^* - R \leq \mathcal{F}_* + R$$

This is an upper semicontinuous function bounded above by a lower semicontinuous function. Thus by the Katetov-Tong insertion theorem[2] there is a continuous function g with

$$\mathcal{F}^* - R \leq g \leq \mathcal{F}_* + R$$

But then for $f \in \mathcal{F}$ we have $f \leq \mathcal{F}^* \leq g + r$ and $f \geq \mathcal{F}_* \geq g - R$. Hence $f \in \overline{B}(g, R)$. Thus $\mathcal{F} \subseteq \overline{B}(g, R)$ and $r(\mathcal{F}) \leq R$. ☐

Lemma 27. *If X is first countable then for every $x \in X$ there is a countable $\mathcal{G} \subseteq \mathcal{F}$ such that $\mathcal{G}^*(x) = \mathcal{F}^*(x)$ and $\mathcal{G}_*(x) = \mathcal{F}_*(x)$.*

Proof. It suffices to show that we can find countable $\mathcal{G}$ such that $\mathcal{G}^*(x) = \mathcal{F}^*(x)$. The same argument will work for finding a $\mathcal{G}$ with $\mathcal{G}_*(x) = \mathcal{F}_*(x)$, and the union of the two will then work for the conclusion.

First note that for any open set U we may find countable $\mathcal{G}$ such that $\sup \mathcal{F}(U) = \sup \mathcal{G}(U)$: Pick $g_n \in \mathcal{F}$ such that $\sup g_n(U) > \sup \mathcal{F}(U) - \frac{1}{n}$ and then take $\{g_n\}$.

Now we may find open sets $U_n \ni x$ forming a neighbourhood basis for x. Then $\inf_n \sup \mathcal{F}(U_n) = \mathcal{F}^*(x)$. Pick countable $\mathcal{G}_n$ such that $\sup \mathcal{F}(U_n) = \sup \mathcal{G}_n(U_n)$. Then $\mathcal{G} = \bigcup_n \mathcal{G}_n$ clearly must have $\mathcal{G}^*(x) = \mathcal{F}^*(x)$.

□

Theorem 28. *Let X be normal and first countable and $\mathcal{F} \subseteq C(X)$ be bounded. Then*

$$r(\mathcal{F}) = \sup\{r(\mathcal{G}) : \mathcal{G} \subseteq \mathcal{F},\ \textit{countable}\ \}$$

i.e. $C(X)$ has ω_1-determined radii.

Proof. Let $R = \sup\{r(\mathcal{G}) : \mathcal{G} \subseteq \mathcal{F},\ \text{countable}\ \}$.

It is clear that $r(\mathcal{F}) \geq R$. We need only show the other direction.

For $x \in X$ we can find countable $\mathcal{G}$ such that $\mathcal{G}^*(x) = \mathcal{F}^*(x)$ and $\mathcal{G}_*(x) = \mathcal{F}_*(x)$. Thus $\mathcal{F}^*(x) - \mathcal{F}_*(x) = \mathcal{G}^*(x) - \mathcal{G}_*(x) \leq 2r(\mathcal{G}) \leq 2R$.

Thus by normality of X, $r(\mathcal{F}) = \frac{1}{2}\sup_x(\mathcal{F}^*(x) - \mathcal{F}_*(x)) \leq R$ □

Corollary 28.1. *If X is normal and first countable then $C(X)$ has no bad sequences.*

Corollary 28.2. *$C(\omega_1)$ has no bad sequences.*

We now have two counterexamples to the converse of 13: Neither $\omega_1 \times (\omega_1 + 1)$ nor $\omega_1 \times \beta\mathbb{N}$ are normal, but ω_1 is $OCA(C(\omega_1 + 1))$ and $C(\beta\mathbb{N})$.

Examining this proof also lets us construct a space X such that $\neg OCA(\omega_1, C(X))$:

Example 29. *Let $A = \{a_\alpha : \alpha \leq \omega_1\}$ and $B = \{b_\alpha : \alpha \leq \omega_1\}$ be two disjoint copies of $\omega_1 + 1$. Let X be the quotient of their union defined by identifying a_{ω_1} and $b_{\omega 1}$.*

For $\alpha < \omega_1$ define $f_\alpha \in C(X)$ as

$$\begin{aligned} f_\alpha(a_\beta) &= -2 && (\beta \leq \alpha) \\ f_\alpha(b_\beta) &= 2 && (\beta \leq \alpha) \\ f_\alpha(x) &= 0 && (\textit{otherwise}) \end{aligned}$$

Then f_α *is a bad sequence in* $C(X)$. *The initial segments have radius* 1, *as* $\{f_\beta : \beta \leq \alpha\} \subseteq B(\frac{1}{2}f_\alpha, 1)$, *but every neighbourhood of* a_{ω_1} *contains the values* -2 *and* 2, *so the tails all have radius* 2.

References

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